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NONEXISTENCE RESULTS OF SOLUTIONS OF SEMILINEAR DIFFERENTIAL INEQUALITIES WITH TEMPERAL FRACTIONAL DERIVATIVE ON THE HEINSENBERG GROUP

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Abstract

Denoting by $\mathbf{D}^{\alpha}_{0|t}$ the time-fractional derivative of order α ($\alpha \in (0,1)$) in the sense of Caputo, and by $\Delta_{\mathbb{H}}$ the Laplacian operator on the $(2N+1)$-dimensional Heisenberg group $\mathbb{H}^N$, we prove some nonexistence results for solutions to problems of the type

$$\mathbf{D}^{\alpha}_{0|t} u - \Delta_{\mathbb{H}}(a\, u) \geqslant |u|^p,$$

$$\begin{cases} \mathbf{D}^{\alpha}_{0|t} u - \Delta_{\mathbb{H}}(a\, u) \geqslant |v|^p, \\ \mathbf{D}^{\delta}_{0|t} v - \Delta_{\mathbb{H}}(b\, v) \geqslant |u|^q, \end{cases}$$

in $\mathbb{H}^N \times \mathbb{R}^+$, with $a,\, b \in L^\infty(\mathbb{H}^N \times \mathbb{R}^+)$.

For $\alpha = 1$ (and $\delta = 1$ in the case of two inequalities), we retrieve the results obtained by Pohozaev-Véron [10] and El Hamidi-Kirane [3] corresponding, respectively, to the parabolic inequalities and parabolic system.

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1. Introduction

Let us begin this section by recalling some known facts about the time-fractional derivative $\mathbf{D}^{\alpha}_{0|t}$, the Heisenberg group $\mathbb{H}^N$ and the operator $\Delta_{\mathbb{H}}$ which will be useful later on.

The left-sided derivative and the right-sided derivative in the sense of Riemann-Liouville for $\psi \in L^1(0,T)$, $\alpha \in (0,1)$, are defined respectively as follows:

$$\left(D^{\alpha}_{0|t}\psi\right)(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{\psi(\sigma)}{(t-\sigma)^{\alpha}}\,d\sigma,$$

$$\left(D^{\alpha}_{t|T}\psi\right)(t) = -\frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_t^T \frac{\psi(\sigma)}{(\sigma-t)^{\alpha}}\,d\sigma,$$

where Γ is the Euler gamma function.

For $\psi' \in L^1(0,T)$, the Caputo derivative of order $\alpha \in (0,1)$ is defined by

$$\left(\mathbf{D}^{\alpha}_{0|t}\psi\right)(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\psi'(\sigma)}{(t-\sigma)^{\alpha}}\,d\sigma,$$

and it is related to the Riemann-Liouville derivative by

$$\mathbf{D}^{\alpha}_{0|t}\psi(t) = D^{\alpha}_{0|t}\big\{\psi(t)-\psi(0)\big\}.$$

Recall also the integration by parts formula

$$\int_0^T \varphi(t)\,\left(D^{\alpha}_{0|t}\psi\right)(t)\,dt = \int_0^T \left(D^{\alpha}_{t|T}\varphi\right)(t)\,\psi(t)\,dt.$$

For more details concerning fractional derivatives, one can refer to books as [9], [11].

The $(2N+1)$-dimensional Heisenberg group $\mathbb{H}^N$ is the space

$$\mathbb{R}^{2N+1} = \big\{\eta = (x,y,\tau) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}\big\},$$

equipped with the group operation "$\circ$" defined by

$$\eta \circ \tilde{\eta} = \left(x+\tilde{x},\, y+\tilde{y},\, \tau+\tilde{\tau}+2\sum_{i=1}^{N}\big(x_i\tilde{y}_i - \tilde{x}_i y_i\big)\right), \tag{1.1}$$

where

$$\eta = (x,y,\tau) = \big(x_1,...,x_N,y_1,...,y_N,\tau\big),$$

$$\tilde{\eta} = (\tilde{x},\tilde{y},\tilde{\tau}) = \big(\tilde{x}_1,...,\tilde{x}_N,\tilde{y}_1,...,\tilde{y}_N,\tilde{\tau}\big).$$

This group multiplication endows $\mathbb{H}^N$ with a structure of a Lie group. The subelliptic Laplacian $\Delta_{\mathbb{H}}$ over $\mathbb{H}^N$ is obtained from the vector fields

$$X_i = \frac{\partial}{\partial x_i} + 2y_i\frac{\partial}{\partial \tau} \quad \text{and} \quad Y_i = \frac{\partial}{\partial y_i} - 2x_i\frac{\partial}{\partial \tau}$$

by the following

$$\Delta_{\mathbb{H}} = \sum_{i=1}^{N} \big(X_i^2 + Y_i^2\big). \tag{1.2}$$

An explicit computation gives us the expression

$$\Delta_{\mathbb{H}} = \sum_{i=1}^{N} \left(\frac{\partial^2}{\partial x_i^2} + \frac{\partial^2}{\partial y_i^2} + 4y_i \frac{\partial^2}{\partial x_i \partial \tau} - 4x_i \frac{\partial^2}{\partial y_i \partial \tau} + 4(x_i^2 + y_i^2) \frac{\partial^2}{\partial \tau^2} \right). \tag{1.3}$$

The operator $\Delta_{\mathbb{H}}$ is a degenerate elliptic operator satisfying the Hörmander condition of order 1 (see [5]). It is invariant with respect to the left multiplication in the group since

$$\Delta_{\mathbb{H}}\Big(u\big(\eta \circ \tilde{\eta}\big)\Big) = \big(\Delta_{\mathbb{H}} u\big)\big(\eta \circ \tilde{\eta}\big), \quad \forall \big(\eta, \tilde{\eta}\big) \in \mathbb{H}^N \times \mathbb{H}^N.$$

An intrinsic distance of the point η to the origin can be defined on $\mathbb{H}^N$ by setting

$$\big|\eta\big|_{\mathbb{H}} = \left(\tau^2 + \sum_{i=1}^{N} \big(x_i^2 + y_i^2\big)^2 \right)^{1/4}.$$

It is also important to observe that $\eta \to |\eta|_{\mathbb{H}}$ is homogeneous of degree one with respect to the natural group of dilatations

$$\delta_\lambda(\eta) = \big(\lambda x, \lambda y, \lambda^2 t\big). \tag{1.4}$$

Remark also, by virtue of (1.3), that the operator $\Delta_{\mathbb{H}}$ is homogeneous of degree 2 with respect to the dilatation δ_λ defined in (1.4), namely

$$\Delta_{\mathbb{H}} = \lambda^2\, \delta_\lambda\big(\Delta_{\mathbb{H}}\big).$$

Concerning the action of $\Delta_{\mathbb{H}}$ on functions u depending only on $\rho := \big|\eta\big|_{\mathbb{H}}$, it is easy to show that

$$\Delta_{\mathbb{H}} u(\rho) = a(\eta) \left(\frac{d^2 u}{d\rho^2} + \frac{Q-1}{\rho} \frac{du}{d\rho} \right),$$

where the function a is defined by

$$a(\eta) = \sum_{i=1}^{N} \frac{x_i^2 + y_i^2}{\rho^2} \quad \text{and} \quad Q = 2N + 2.$$

This last number Q is called the homogeneous dimension of $\mathbb{H}^N$.

We escort the paper of Kirane, Laskri and Tatar [7] but taking into account of the Laplacian on the Heisenberg group. The critical exponent found here is the same determined in [7] for the case of a single equation as well as for a system of two equations; of course, with some appropriate modification coming from the replacement of the fractional power of the Laplacian by the Laplacian on the Heisenberg group.

Nonexistence results for solutions of semilinear inequalities of the type

$$\frac{\partial u}{\partial t} - \Delta_{\mathbb{H}}(au) \geqslant |\eta|_{\mathbb{H}}^{\gamma}|u|^p \tag{1.5}$$

were studied by Pohozaev and Véron [10], and they proved that no weak solution u exists provided

$$\int_{\mathbb{R}^{2N+1}} u(\eta,0)\, d\eta \geqslant 0, \quad \gamma > -2 \quad \text{and} \qquad 1 < p \leqslant p_c := \frac{Q+2+\gamma}{Q}. \tag{1.6}$$

In this paper, we generalize this result to evolution inequalities with temperal fractional derivative. More precisely, let a be a bounded and measurable function defined in $\mathbb{H}^N \times \mathbb{R}^+$, then we prove that there exists no locally integrable function u defined in whole $\mathbb{H}^N \times \mathbb{R}^+$, such that $u \in L^p_{loc}\left(\mathbb{H}^N \times \mathbb{R}^+, |\eta|_{\mathbb{H}}^{\gamma} d\eta dt\right)$, satisfying

$$\mathbf{D}^{\alpha}_{0|t}(u) - \Delta_{\mathbb{H}}(au) \geqslant |\eta|_{\mathbb{H}}^{\gamma}|u|^p \tag{1.7}$$

whenever $1 < p < \dfrac{Q+\frac{2}{\alpha}+\gamma}{Q+2\left(\frac{1}{\alpha}-1\right)}$, for an arbitrary $\alpha \in (0,1)$. And thus, for $\alpha = 1$, we retrieve the critical exponent introduced in (1.6).

In [3], El Hamidi and Kirane presented similar results for system of m parabolic semilinear inequalities. Their results have been generalized, by El Hamidi and Obeid [4], to systems of m-semilinear inequalities with higher-order time derivative.

Concerning nonexistence results to elliptic semilinear inequalities and systems, one can refer to [1], [2], [6] and [8].

2. Case of single inequality

We first consider single inequality of the following type

$$\begin{cases} \mathbf{D}^{\alpha}_{0|t} u - \Delta_{\mathbb{H}}(au) & \geqslant & |\eta|^{\gamma}_{\mathbb{H}} |u|^p, \quad \text{for} \quad (\eta, t) \in \mathbb{H}^N \times \mathbb{R}^+, \\ u(\eta, 0) & = & u_0(\eta) \geqslant 0, \quad \text{for} \quad \eta \in \mathbb{H}^N, \end{cases} \tag{2.1}$$

where $a = a(\eta, t)$ is a bounded and measurable function defined in $\mathbb{R}^{2N+1} \times \mathbb{R}^+$, γ and $p > 1$ are two real numbers. We identify points in $\mathbb{H}^N$ with points in $\mathbb{R}^{2N+1}$. We also recall that the natural Haar measure in $\mathbb{H}^N$ is identical to the Lebesgue measure $d\eta = dxdyd\tau$ in $\mathbb{R}^{2N+1} = \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$.

Definition 2.1. A local weak solution u of the differential inequality (2.1) in Q_T ($Q_T := \mathbb{R}^{2N+1} \times (0, T)$) with positive initial data $u_0 \in L^1_{loc}(\mathbb{R}^{2N+1})$ is a locally integrable function such that $u \in L^p\left(Q_T; |\eta|^{\gamma}_{\mathbb{H}} d\eta dt\right)$ which satisfies

$$\int_{Q_T} \left(-u\, D^{\alpha}_{t|T}\varphi + a\, u\, \Delta_{\mathbb{H}}\varphi + |\eta|^{\gamma}_{\mathbb{H}} |u|^p \varphi + u_0\, D^{\alpha}_{t|T}\varphi \right) d\eta dt \leqslant 0, \tag{2.2}$$

for any nonnegative test function $\varphi \in C^{2,1}_{\eta,t}(Q_T)$ verifying $\varphi(., T) = 0$.

The integrals in the above definition are supposed to be convergent. If in the definition $T = +\infty$, the solution is called global.

Theorem 2.1. *Let $N \geqslant 1$ and $p > 1$. Assume*

$$1 < p < \frac{Q + \frac{2}{\alpha} + \gamma}{Q + 2\left(\frac{1}{\alpha} - 1\right)}, \tag{2.3}$$

then there exists no global weak solution of (2.1) *other than the trivial one.*

P r o o f. The method is based on a suitable choice of the test function. We assume that problem (2.1) has a nontrivial global weak solution, namely u. Let T, R and $\theta > 1$ (will be determined later) be three positive reals and let φ be a smooth nonnegative test function. Since the initial data u_0 is nonnegative, the variational formulation (2.2) implies

$$\int_{Q_{TR^{4/\theta}}} |\eta|^{\gamma}_{\mathbb{H}} |u|^p \varphi\, d\eta dt \leqslant \int_{Q_{TR^{4/\theta}}} u\, D^{\alpha}_{t|TR^{4/\theta}}\varphi\, d\eta dt - \int_{Q_{TR^{4/\theta}}} a\, u\, \Delta_{\mathbb{H}}\varphi\, d\eta dt. \tag{2.4}$$

The test function φ is chosen so that

$$\int_{Q_{TR^{4/\theta}}} \left\{ \left| D^{\alpha}_{t|T}\varphi \right|^{p'} + \left| \Delta_{\mathbb{H}}\varphi \right|^{p'} \right\} \left(|\eta|^{\gamma}_{\mathbb{H}}\varphi \right)^{-p'/p} d\eta dt < \infty.$$

In order to estimate the right-hand side of (2.4), we apply Young's inequality for an arbitrary $\varepsilon > 0$. It follows

$$\left\{ \begin{aligned} \int_{Q_{TR^{4/\theta}}} u\, D^{\alpha}_{t|TR^{4/\theta}}\varphi \, d\eta dt \;\; &\leqslant \;\; \varepsilon \int_{Q_{TR^{4/\theta}}} |\eta|^{\gamma}_{\mathbb{H}} |u|^{p}\, \varphi \, d\eta dt \\ &\quad + C_{\varepsilon} \int_{Q_{TR^{4/\theta}}} \left| D^{\alpha}_{t|TR^{4/\theta}}\varphi \right|^{p'} \left(|\eta|^{\gamma}_{\mathbb{H}}\varphi \right)^{\frac{-p'}{p}} d\eta dt, \end{aligned} \right.$$

and

$$\left\{ \begin{aligned} \int_{Q_{TR^{4/\theta}}} a\, u\, \Delta_{\mathbb{H}}\varphi \, d\eta dt \;\; &\leqslant \;\; \varepsilon \int_{Q_{TR^{4/\theta}}} |\eta|^{\gamma}_{\mathbb{H}} |u|^{p}\, \varphi \, d\eta dt \\ &\quad + C_{\varepsilon} \, \|a\|^{p'}_{L^{\infty}} \int_{Q_{TR^{4/\theta}}} |\Delta_{\mathbb{H}}\varphi|^{p'} \left(|\eta|^{\gamma}_{\mathbb{H}}\varphi \right)^{\frac{-p'}{p}} d\eta dt. \end{aligned} \right.$$

Choosing ε small enough, we get

$$\int_{Q_{TR^{4/\theta}}} |\eta|^{\gamma}_{\mathbb{H}} |u|^{p}\varphi \leqslant C_{\varepsilon} \int_{Q_{TR^{4/\theta}}} \left\{ \left| D^{\alpha}_{t|TR^{4/\theta}}\varphi \right|^{p'} + |\Delta_{\mathbb{H}}\varphi|^{p'} \right\} \left(|\eta|^{\gamma}_{\mathbb{H}}\varphi \right)^{\frac{-p'}{p}} . \tag{2.5}$$

Now we take

$$\varphi(\eta, t) \, = \, \varphi\big(x, y, \tau, t\big) \, = \, \Phi\left(\frac{\tau^2 + |x|^4 + |y|^4 + t^{\theta}}{R^4} \right),$$

where $\Phi \in \mathscr{D}(\mathbb{R}^+)$ satisfies $0 \leqslant \Phi \leqslant 1$ and

$$\Phi(r) \, = \, \begin{cases} 0 & \text{if} \quad r \geqslant 2, \\ 1 & \text{if} \quad 0 \leqslant r \leqslant 1. \end{cases} \tag{2.6}$$

Then

$$\left\{ \begin{aligned} &\Delta_{\mathbb{H}}\varphi\,(\eta, t) \, = \, \frac{4(N+4)\,\Phi'(\rho)}{R^4} \Big[|x|^2 + |y|^2 \Big] \\ &\quad + \frac{16\,\Phi''(\rho)}{R^8} \Bigg[\big(|x|^6 + |y|^6 \big) + \tau^2 \big(|x|^2 + |y|^2 \big) + 2\,\tau\, \langle x, y \rangle \big(|x|^2 - |y|^2 \big) \Bigg], \end{aligned} \right. \tag{2.7}$$

where

$$\rho \;=\; \frac{\tau^2+|x|^4+|y|^4+t^\theta}{R^4}.$$

In order to estimate the right-hand side of (2.7), we perform the change of variables

$$\tilde{t}=R^{-4/\theta}t,\quad \tilde{\tau}=R^{-2}\tau,\quad \tilde{x}=R^{-1}x,\quad \tilde{y}=R^{-1}y.$$

Let

$$\tilde{\rho} \;=\; \tilde{\tau}^2+|\tilde{x}|^4+|\tilde{y}|^4+\tilde{t}^{\,\theta},$$

and

$$\Omega \;=\; \left\{(\tilde{\eta},\tilde{t})=(\tilde{x},\tilde{y},\tilde{\tau},\tilde{t})\in\mathbb{R}^{2N+1}\times\mathbb{R}^+,\quad \tilde{\tau}^2+|\tilde{x}|^4+|\tilde{y}|^4+\tilde{t}^{\,\theta}\;\leqslant\;2\right\}.$$

Then

$$\left|\Delta_{\mathbb{H}}\varphi\,(\tilde{\eta},\tilde{t})\right| \;\leqslant\; \frac{C}{R^2},\quad \forall\,(\tilde{\eta},\tilde{t})\in\Omega. \tag{2.8}$$

Since $d\eta dt=R^{2N+2+4/\theta}\,d\tilde{\eta}d\tilde{t}$ and $\left|\eta\right|_{\mathbb{H}}=R\left|\tilde{\eta}\right|_{\mathbb{H}}$, we derive from (2.8) that

$$\left\{\begin{array}{l}\displaystyle\int_{Q_{TR^{4/\theta}}}\left|\Delta_{\mathbb{H}}\varphi\right|^{p'}\left(|\eta|_{\mathbb{H}}^{\gamma}\varphi\right)^{-p'/p}\\ \qquad\qquad\displaystyle\leqslant\; R^{-2p'+2N+2+\frac{4}{\theta}-\gamma\frac{p'}{p}}\int_{\Omega}\left|\Delta_{\mathbb{H}}\Phi\circ\tilde{\rho}\right|^{p'}\left(\left|\tilde{\eta}\right|_{\mathbb{H}}^{\gamma}\Phi\circ\tilde{\rho}\right)^{-p'/p}d\tilde{\eta}d\tilde{t}.\end{array}\right.$$

Taking into account the definition of the time derivative in the sense of Riemann-Liouville, we obtain

$$\left\{\begin{array}{l}\displaystyle\int_{Q_{TR^{4/\theta}}}\left|D^{\alpha}_{t|TR^{4/\theta}}\varphi\right|^{p'}\left(|\eta|_{\mathbb{H}}^{\gamma}\varphi\right)^{-p'/p}\\ \qquad\qquad\displaystyle\leqslant\; R^{-\frac{4\alpha}{\theta}p'+2N+2+\frac{4}{\theta}-\gamma\frac{p'}{p}}\int_{\Omega}\left|D^{\alpha}_{\tilde{t}|T}\Phi\circ\tilde{\rho}\right|^{p'}\left(\left|\tilde{\eta}\right|_{\mathbb{H}}^{\gamma}\Phi\circ\tilde{\rho}\right)^{-p'/p}d\tilde{\eta}d\tilde{t}.\end{array}\right.$$

In order to get the same exponent in R, we take $\theta=2\alpha$ and deduce

$$\int_{Q_{TR^{2/\alpha}}}|\eta|_{\mathbb{H}}^{\gamma}|u|^{p}\varphi\,d\eta dt\;\leqslant\;C\,R^{-2p'+2N+2+\frac{2}{\alpha}-\gamma\frac{p'}{p}}, \tag{2.9}$$

where

$$C=C_{\varepsilon}\int_{\Omega}\left\{\left|D^{\alpha}_{\tilde{t}|T}\Phi\circ\tilde{\rho}\right|^{p'}+\left|\Delta_{\mathbb{H}}\Phi\circ\tilde{\rho}\right|^{p'}\right\}\left(|\tilde{\eta}|_{\mathbb{H}}^{\gamma}\Phi\circ\tilde{\rho}\right)^{\frac{-p'}{p}}\,d\tilde{\eta}d\tilde{t}.$$

Now, in the case where

$$1 < p < \frac{2N+2+\gamma+\frac{2}{\alpha}}{2N+\frac{2}{\alpha}} = \frac{Q+\frac{2}{\alpha}+\gamma}{Q+2\left(\frac{1}{\alpha}-1\right)},$$

the exponent of R in (2.9) is negative. Letting R go to infinity and using Fatou's lemma, we deduce that

$$\int_{\mathbb{R}^+}\int_{\mathbb{R}^{2N+1}} |\eta|_{\mathbb{H}}^{\gamma}|u|^p d\eta dt \;=\; 0, \tag{2.10}$$

which implies that $u \equiv 0$. This contradicts the fact that u is a nontrivial weak solution of (2.1), which achieves the proof. ∎

Remark 2.1. When $\alpha = 1$, we recover the case of parabolic inequality of the type

$$\frac{\partial u}{\partial t} - \Delta_{\mathbb{H}}(au) \;\geqslant\; |\eta|_{\mathbb{H}}^{\gamma}|u|^p, \tag{2.11}$$

studied by Pohozaev and Véron in [10].

3. System of two inequalities

We consider the following system:

$$\begin{cases} \mathbf{D}^{\alpha}_{0|t}u - \Delta_{\mathbb{H}}\big(a\,u\big) & \geqslant \quad |\eta|_{\mathbb{H}}^{\beta}\,|v|^p, \quad \text{in} \quad \mathbb{H}^N \times \mathbb{R}^+, \\ \mathbf{D}^{\delta}_{0|t}v - \Delta_{\mathbb{H}}\big(b\,v\big) & \geqslant \quad |\eta|_{\mathbb{H}}^{\gamma}\,|u|^q, \quad \text{in} \quad \mathbb{H}^N \times \mathbb{R}^+, \end{cases} \tag{3.1}$$

where $\mathbf{D}^{\alpha}_{0|t}$ (resp. $\mathbf{D}^{\delta}_{0|t}$) denotes the time-fractional derivative of order α, $\alpha \in (0,1)$ (resp. δ, $\delta \in (0,1)$), in the sense of Caputo.

The functions a and b introduced in (3.1) are supposed to be measurable and bounded functions on $\mathbb{H}^N \times \mathbb{R}^+$. While the exponents p, $q > 1$ and β, γ are real numbers.

Denoting by $D^{\alpha}_{0|t}$ (resp. $D^{\delta}_{0|t}$) the time-fractional derivative of order α (resp. δ) in the sense of Riemann-Liouville, we adopt the following

Definition 3.1. A local weak solution $\big(u, v\big)$ of the system (3.1) in $Q_T := \mathbb{R}^{2N+1} \times (0,T)$ with positive initial conditions $\big(u_0, v_0\big) \in L^1_{loc}(\mathbb{R}^{2N+1})$ is a couple of locally integrable functions $\big(u, v\big)$ such that

$$\big(u, v\big) \;\in\; L^q_{loc}\Big(Q_T, |\eta|_{\mathbb{H}}^{\gamma} d\eta dt\Big) \times L^p_{loc}\Big(Q_T, |\eta|_{\mathbb{H}}^{\beta} d\eta dt\Big)$$

satisfying

$$\begin{cases} \displaystyle\int_{Q_T} \Big(-u\, D^{\alpha}_{t|T}\varphi \,+\, a\,u\,\Delta_{\mathbb{H}}\varphi \,+\, |\eta|^{\beta}_{\mathbb{H}}|v|^{p}\varphi \,+\, u_0\, D^{\alpha}_{t|T}\varphi\Big) d\eta dt \;\leqslant\; 0, \\ \displaystyle\int_{Q_T} \Big(-v\, D^{\delta}_{t|T}\varphi \,+\, b\,v\,\Delta_{\mathbb{H}}\varphi \,+\, |\eta|^{\gamma}_{\mathbb{H}}|u|^{q}\varphi \,+\, v_0\, D^{\delta}_{t|T}\varphi\Big) d\eta dt \;\leqslant\; 0, \end{cases} \tag{3.2}$$

for any nonnegative test function $\varphi \in C^{2,1}_{\eta,t}\Big(Q_T\Big)$, such that $\varphi(.,T) = 0$.

As in Definition 2.1, we assume that the integrals in (3.2) are convergent. If in the definition $T = +\infty$, the solution is called global.

THEOREM 3.1. *Assume that*

$$Q < Q^{\star}_e := \max\Big\{Q_1, Q_2\Big\},$$

where

$$\begin{cases} Q_1 = \dfrac{\alpha\Big(1+\frac{\gamma}{2q}\Big) + \frac{\delta}{q}\Big(1+\frac{\beta}{2p}\Big) - \Big(1-\frac{1}{pq}\Big)}{\frac{\alpha}{2q'} + \frac{\delta}{2p'q}}, \\ Q_2 = \dfrac{\delta\Big(1+\frac{\beta}{2p}\Big) + \frac{\alpha}{p}\Big(1+\frac{\gamma}{2q}\Big) - \Big(1-\frac{1}{pq}\Big)}{\frac{\delta}{2p'} + \frac{\alpha}{2q'p}}. \end{cases} \tag{3.3}$$

Then, there exists no nontrivial global weak solution (u, v) *of system* (3.1).

P r o o f. As in the proof of Theorem 1, we argue by contradiction. Suppose that (u, v) is a nontrivial weak solution which exists globally in time. That is (u, v) exists in $(0, T^{\star})$ for an arbitrary $T^{\star} > 0$.

Let T and R be two positive real numbers such that $0 < TR < T^{\star}$.

Since initial conditions u_0 and v_0 are nonnegative, the variational formulation (3.2) implies

$$\begin{cases} \displaystyle\int_{Q_{TR}} |\eta|^{\beta}_{\mathbb{H}}|v|^{p}\varphi\, d\eta dt \;\leqslant\; \int_{Q_{TR}} u\, D^{\alpha}_{t|TR}\varphi\, d\eta dt \;-\; \int_{Q_{TR}} a\,u\,\Delta_{\mathbb{H}}\varphi\, d\eta dt, \\ \displaystyle\int_{Q_{TR}} |\eta|^{\gamma}_{\mathbb{H}}|u|^{q}\varphi\, d\eta dt \;\leqslant\; \int_{Q_{TR}} v\, D^{\delta}_{t|TR}\varphi\, d\eta dt \;-\; \int_{Q_{TR}} b\,v\,\Delta_{\mathbb{H}}\varphi\, d\eta dt. \end{cases} \tag{3.4}$$

Using Hölder's inequality, it follows

$$\begin{cases} \displaystyle\int_{Q_{TR}} |\eta|^{\beta}_{\mathbb{H}}|v|^{p}\varphi \;\leqslant\; \Big(\int_{Q_{TR}} |u|^{q}|\eta|^{\gamma}_{\mathbb{H}}\varphi\Big)^{\frac{1}{q}} \Big(\int_{Q_{TR}} \big|D^{\alpha}_{t|TR}\varphi\big|^{q'} \big(|\eta|^{\gamma}_{\mathbb{H}}\varphi\big)^{-\frac{q'}{q}}\Big)^{\frac{1}{q'}} \\ \qquad\qquad + \|a\|_{L^{\infty}} \Big(\int_{Q_{TR}} |u|^{q}|\eta|^{\gamma}_{\mathbb{H}}\varphi\Big)^{\frac{1}{q}} \Big(\int_{Q_{TR}} \big|\Delta_{\mathbb{H}}\,\varphi\big|^{q'} \big(|\eta|^{\gamma}_{\mathbb{H}}\varphi\big)^{-\frac{q'}{q}}\Big)^{\frac{1}{q'}}, \end{cases}$$

and

$$\left\{\begin{aligned} \int_{Q_{TR}} |\eta|_{\mathbb{H}}^{\gamma} |u|^q \varphi \leqslant \left(\int_{Q_{TR}} |v|^p |\eta|_{\mathbb{H}}^{\beta} \varphi\right)^{\frac{1}{p}} \left(\int_{Q_{TR}} \left|D_{t|TR}^{\delta} \varphi\right|^{q'} \left(|\eta|_{\mathbb{H}}^{\beta}\varphi\right)^{-\frac{p'}{p}}\right)^{\frac{1}{p'}} \\ + \|b\|_{L^\infty} \left(\int_{Q_{TR}} |v|^p |\eta|_{\mathbb{H}}^{\beta} \varphi\right)^{\frac{1}{p}} \left(\int_{Q_{TR}} \left|\Delta_{\mathbb{H}}\, \varphi\right|^{p'} \left(|\eta|_{\mathbb{H}}^{\beta}\varphi\right)^{-\frac{p'}{p}}\right)^{\frac{1}{p'}}. \end{aligned}\right.$$

In the sequel, C denotes a constant which may vary from line to line but is independent of the terms which will take part in any limit processing. So we have

$$\int_{Q_{TR}} |\eta|_{\mathbb{H}}^{\beta} |v|^p \varphi \leqslant C \left(\int_{Q_{TR}} |u|^q |\eta|_{\mathbb{H}}^{\gamma} \varphi\right)^{\frac{1}{q}} \mathscr{A}, \tag{3.5}$$

and

$$\int_{Q_{TR}} |\eta|_{\mathbb{H}}^{\gamma} |u|^q \varphi \leqslant C \left(\int_{Q_{TR}} |v|^p |\eta|_{\mathbb{H}}^{\beta} \varphi\right)^{\frac{1}{p}} \mathscr{B}, \tag{3.6}$$

where

$$\mathscr{A} := \left(\int_{Q_{TR}} \left|D_{t|TR}^{\alpha} \varphi\right|^{q'} \left(|\eta|_{\mathbb{H}}^{\gamma}\varphi\right)^{-\frac{q'}{q}}\right)^{\frac{1}{q'}} + \left(\int_{Q_{TR}} \left|\Delta_{\mathbb{H}}\, \varphi\right|^{q'} \left(|\eta|_{\mathbb{H}}^{\gamma}\varphi\right)^{-\frac{q'}{q}}\right)^{\frac{1}{q'}},$$

$$\mathscr{B} := \left(\int_{Q_{TR}} \left|D_{t|TR}^{\delta} \varphi\right|^{p'} \left(|\eta|_{\mathbb{H}}^{\beta}\varphi\right)^{-\frac{p'}{p}}\right)^{\frac{1}{p'}} + \left(\int_{Q_{TR}} \left|\Delta_{\mathbb{H}}\, \varphi\right|^{p'} \left(|\eta|_{\mathbb{H}}^{\beta}\varphi\right)^{-\frac{p'}{p}}\right)^{\frac{1}{p'}}.$$

Combining (3.5) and (3.6) we obtain

$$\left(\int_{Q_{TR}} |\eta|_{\mathbb{H}}^{\beta} |v|^p \varphi\right)^{1-\frac{1}{pq}} \leqslant C\, \mathscr{B}^{\frac{1}{q}}\, \mathscr{A}, \tag{3.7}$$

and

$$\left(\int_{Q_{TR}} |\eta|_{\mathbb{H}}^{\gamma} |u|^q \varphi\right)^{1-\frac{1}{pq}} \leqslant C\, \mathscr{A}^{\frac{1}{p}}\, \mathscr{B}. \tag{3.8}$$

Now we take

$$\varphi(\eta, t) = \varphi(x, y, \tau, t) = \Phi\left(\frac{\tau^{2\theta_j} + |x|^{4\theta_j} + |y|^{4\theta_j} + t^4}{R^4}\right), \quad j = 1, 2, \tag{3.9}$$

where Φ is a smooth nonnegative nonincreasing function satisfying (2.6) and $\theta_j > 1$, $j = 1, 2$, will be determined later on.

Then,

$$\begin{cases} \Delta_{\mathbb{H}}\,\varphi(\eta,t) \;=\; \dfrac{16\,\theta_j^2\,\Phi''(\rho)}{R^8}\Bigg[|x|^{2(4\theta_j-1)}+|y|^{2(4\theta_j-1)} \\ \qquad +2\,\tau^{2\theta_j-1}\,\langle x,y\rangle\Big(|x|^{2(2\theta_j-1)}-|y|^{2(2\theta_j-1)}\Big)+\;\tau^{2(2\theta_j-1)}\Big(|x|^2+|y|^2\Big)\Bigg] \\ \qquad +\;\dfrac{4\,\theta_j\,\Phi'(\rho)}{R^4}\Bigg[\Big(N+2(2\theta_j-1)\Big)\Big(|x|^{2(2\theta_j-1)}+|x|^{2(2\theta_j-1)}\Big) \\ \qquad +2(2\theta_j-1)\,\tau^{2(\theta_j-1)}\Big(|x|^2+|y|^2\Big)\Bigg]. \end{cases}$$

where $\rho=\dfrac{\tau^{2\theta_j}+|x|^{4\theta_j}+|y|^{4\theta_j}+t^4}{R^4}$.

In order to estimate the right-hand side of the last equality, we apply the scaling: $\big(\eta,t\big)=(x,y,\tau,t)\mapsto\tilde{\eta}=(\tilde{x},\tilde{y},\tilde{\tau},\tilde{t})$

$$\tilde{x}=R^{-\frac{1}{\theta_j}}x,\quad \tilde{y}=R^{-\frac{1}{\theta_j}}y,\quad \tilde{\tau}=R^{-\frac{2}{\theta_j}}\tau,\quad \tilde{t}=R^{-1}t. \tag{3.10}$$

Let

$$\Omega=\Big\{\big(\tilde{\eta},\tilde{t}\big)=\big(\tilde{x},\tilde{y},\tilde{\tau},\tilde{t}\big)\in\mathbb{H}^N\times\mathbb{R}^+\;:\;|\tilde{\tau}|^2+|\tilde{x}|^4+|\tilde{y}|^4+\tilde{t}^{\theta_j}<2\Big\}.$$

Then, it follows that

$$\big|\Delta_{\mathbb{H}}\,\varphi(\tilde{\eta},\tilde{t})\big|\leqslant\frac{C}{R^{\frac{2}{\theta_j}}},\quad \forall\;\big(\tilde{\eta},\tilde{t}\big)\in\Omega. \tag{3.11}$$

Since $\big|\eta\big|_{\mathbb{H}}=R^{\frac{1}{\theta_j}}\big|\tilde{\eta}\big|_{\mathbb{H}}$ and $d\eta dt=R^{4N+1+\frac{2}{\theta_j}}d\tilde{\eta}d\tilde{t}$, we derive the following estimates:

- For $j=1$, we choose θ_1 such that the right-hand sides of

$$\begin{cases} \displaystyle\int_{Q_{TR}}\Big|D^{\alpha}_{t|TR}\varphi\Big|^{q'}\Big(\big|\eta\big|^{\gamma}_{\mathbb{H}}\,\varphi\Big)^{-\frac{q'}{q}}\,d\eta dt \\ \qquad =\;R^{-\alpha q'-\gamma\theta_1\frac{q'}{q}+\frac{2N+2}{\theta_1}+1}\displaystyle\int_{\Omega}\Big|D^{\alpha}_{\tilde{t}|T}\Phi\circ\tilde{\rho}\Big|^{q'}\Big(\big|\tilde{\eta}\big|^{\gamma}_{\mathbb{H}}\,\Phi\circ\tilde{\rho}\Big)^{-\frac{q'}{q}}\,d\tilde{\eta}d\tilde{t} \end{cases}$$

and

$$\begin{cases} \displaystyle\int_{Q_{TR}}|\Delta_{\mathbb{H}}\,\varphi|^{q'}\Big(\big|\eta\big|^{\gamma}_{\mathbb{H}}\,\varphi\Big)^{-\frac{q'}{q}}\,d\eta dt \\ \qquad\leqslant\;C\;R^{-\frac{2}{\theta_1}q'-\gamma\theta_1\frac{q'}{q}+\frac{2N+2}{\theta_1}+1}\displaystyle\int_{\Omega}\Big(\big|\tilde{\eta}\big|^{\gamma}_{\mathbb{H}}\,\Phi\circ\tilde{\rho}\Big)^{-\frac{q'}{q}}\,d\tilde{\eta}d\tilde{t}, \end{cases}$$

are of the same order in R. For this end, we take $\theta_1 = \dfrac{2}{\alpha}$ and we obtain

$$\mathscr{A} \leqslant C\, R^{-\alpha-\frac{\alpha\gamma}{2q}+\frac{\alpha}{2}\,\frac{2N+2}{q'}+\frac{1}{q'}}.$$

- For $j=2$, taking $\theta_2 = \dfrac{2}{\delta}$, we get

$$\mathscr{B} \leqslant C\, R^{-\delta-\frac{\delta\beta}{2p}+\frac{\delta}{2}\,\frac{2N+2}{p'}+\frac{1}{p'}}.$$

And thus, from (3.7) and (3.8), it follows

$$\left(\int_{Q_T} |\eta|_{\mathbb{H}}^{\beta} |v|^p \varphi \, d\eta dt\right)^{1-\frac{1}{pq}} \leqslant C\, R^{-\alpha-\frac{\alpha\gamma}{2q}+\frac{\alpha}{2}\,\frac{2N+2}{q'}+\frac{1}{q'}+\frac{1}{q}\{-\delta p'-\frac{\delta\beta}{2p}+\frac{\delta}{2}\,\frac{2N+2}{p'}+\frac{1}{p'}\}} \tag{3.12}$$

and

$$\left(\int_{Q_T} |\eta|_{\mathbb{H}}^{\gamma} |u|^q \varphi \, d\eta dt\right)^{1-\frac{1}{pq}} \leqslant C\, R^{-\delta-\frac{\delta\beta}{2p}+\frac{\delta}{2}\,\frac{2N+2}{p'}+\frac{1}{p'}+\frac{1}{p}\{-\alpha q'-\frac{\alpha\gamma}{2q}+\frac{\alpha}{2}\,\frac{2N+2}{q'}+\frac{1}{q'}\}}. \tag{3.13}$$

Thus, it suffices to assume

$$\left\{\begin{array}{c} -\alpha - \dfrac{\alpha\gamma}{2q} + \dfrac{\alpha}{2}\dfrac{Q}{q'} + \dfrac{1}{q'} + \dfrac{1}{q}\left\{-\delta p' - \dfrac{\delta\beta}{2p} + \dfrac{\delta}{2}\dfrac{Q}{p'} + \dfrac{1}{p'}\right\} < 0 \\ \text{or} \\ -\delta - \dfrac{\delta\beta}{2p} + \dfrac{\delta}{2}\dfrac{Q}{p'} + \dfrac{1}{p'} + \dfrac{1}{p}\left\{-\alpha q' - \dfrac{\alpha\gamma}{2q} + \dfrac{\alpha}{2}\dfrac{Q}{q'} + \dfrac{1}{q'}\right\} < 0. \end{array}\right.$$

This condition is equivalent to

$$Q < Q_e^{\star} = \max\Big\{Q_1, Q_2\Big\},$$

where Q_1 and Q_2 are defined in (3.3).

Finally, letting $R \to \infty$ and taking into account estimates (3.5) and (3.8) or (3.6) and (3.7) , we obtain

$$\int_{\mathbb{R}^{2N+1}} \int_{\mathbb{R}^+} |\eta|_{\mathbb{H}}^{\beta} |v|^p \, d\eta dt \leqslant 0, \tag{3.14}$$

and

$$\int_{\mathbb{R}^{2N+1}} \int_{\mathbb{R}^+} |\eta|_{\mathbb{H}}^{\gamma} |u|^q \, d\eta dt \leqslant 0. \tag{3.15}$$

We then conclude that $v \equiv 0$ and $u \equiv 0$, which is a contradiction. ■

REMARK 3.1. In [3], El Hamidi and Kirane show that the system

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta_{\mathbb{H}}(a_1 u) \geqslant |\eta|_{\mathbb{H}}^{\gamma_1} |v|^{p_1}, \\ \dfrac{\partial v}{\partial t} - \Delta_{\mathbb{H}}(a_2 v) \geqslant |\eta|_{\mathbb{H}}^{\gamma_2} |u|^{p_2}, \end{cases}$$

does not admit global solution other than the trivial one whenever

$$Q \leqslant Q_e^{\star\star} := \max \left\{ \frac{\gamma_1 + 2 + p_1(\gamma_2 + 2)}{p_1 p_2 - 1}, \frac{\gamma_2 + 2 + p_2(\gamma_1 + 2)}{p_1 p_2 - 1} \right\}.$$

Up to replace γ_1 (resp. γ_2) by β (resp. γ) and p_1 (resp. p_2) by p (resp. q), the critical exponent $Q_e^{\star\star}$ is nothing but the one introduced in Theorem 3.1 for $\alpha = \delta = 1$.

REMARK 3.2. The analysis could be performed for the more general system

$$\begin{cases} \mathbf{D}_{0|t}^{\alpha_i} u_i - \Delta_{\mathbb{H}}(a_i u_i) \geqslant |\eta|_{\mathbb{H}}^{\gamma_{i+1}} |u_{i+1}|^{p_{i+1}}, \quad \eta \in \mathbb{H}^N, \ t \in \mathbb{R}^+, \quad 1 \leqslant i \leqslant m, \\ u_{m+1} = u_1, \end{cases}$$

$m \geqslant 2$, $p_{m+1} = p_1$, $\gamma_{m+1} = \gamma_1$, $\alpha_i \in (0,1)$ and $a_i \in L^\infty\left(\mathbb{H}^N \times \mathbb{R}^+\right)$ for all $i \in \{1, ..., m\}$.

REMARK 3.3. We can also study the following problem

$$\mathbf{D}_{0|t}^{\alpha} u - \Delta_{\mathbb{H}}(au) \geqslant |\eta|_{\mathbb{H}}^{\gamma} |u|^p, \tag{3.16}$$

where $\mathbf{D}_{0|t}^{\alpha}$ denotes the Caputo fractional derivative of order α, $\alpha \in (1,2)$. Therefore, using the same techniques and escorting the scheme of the proof of Theorem 2.1, we retrieve the critical exponent introduced in (2.3). Thus, taking $\alpha = 2$ in (2.3), i.e. when inequality (3.16) reduces to the hyperbolic one

$$\frac{\partial^2 u}{\partial t^2} - \Delta_{\mathbb{H}}(au) \geqslant |\eta|_{\mathbb{H}}^{\gamma} |u|^p, \tag{3.17}$$

we retrieve the critical exponent found by Pohozaev-Véron in [10] namely,

$$p_c := \frac{Q + 1 + \gamma}{Q - 1}.$$

REMARK 3.4. If one let formally α to infinity, the critical exponent introduced in (2.3) becomes

$$p_c^\infty := \frac{Q + \gamma}{Q - 2},$$

which is nothing but the one determined in [10] for the case of elliptic inequality.

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